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Phase Plane Analysis of Dynamically Loaded Journal Bearings[1]

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The dynamic behavior of a perfectly aligned, infinitely long, massless journal is investigated while it is rotating in a complete bearing under the influence of external load and fluid pressure forces. Harrison's classical nonlinear equations describing this condition are analyzed by phase-plane techniques which result in velocity versus displacement plots of the journal eccentricity. It is found that the journal is normally neutrally stable as exhibited by a center singularity in the phase plane. Guided by the phase-plane analysis, it also seems possible to explain the erratic or unstable journal behavior frequently found in journal bearings operating at high rotational speeds and/or light loadings. It is shown that this instability is also possible at slow speeds and/or heavy loading. Examples of phase-plane portraits are presented and these are correlated with the physical journal motion.

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H. T. ALBACHTEN

NOMENCLATURE

c = radial clearance, in.
e = eccentricity, in.
f_1, f_2 = nonlinear perturbations
J = a constant, $\frac{12\pi\mu r^3 L}{Wc^2}$, sec,
($J\omega_1 = 12\pi^2 S$)
L = axial length of the journal, in.
p = fluid pressure psi
r = journal radius, in.
$r_o = (\bar{x}^2 + \bar{y}^2)^{1/2}$
R = bearing radius, in.
S = Sommerfeld number, $\frac{\mu r^3 L \omega_1}{\pi W c^2}$
t = time, sec
W = magnitude of the load on the journal, lbs.
x = phase plane co-ordinate = ϵ
y = phase plane co-ordinate = $J\frac{d\epsilon}{dt}$
$\bar{x}, \bar{y}$ = co-ordinates to a point in phase plane measured from singular point
x^*, y^* = phase plane co-ordinates of a singular point
$X(x,y)$ = denominator of right-hand side of differential equation for integral curves
$Y(x,y)$ = numerator of right-hand side of differential equation for integral curves
$z = x^{*2}$
μ = fluid viscosity, lb sec/in.^{2}
ϕ = attitude angle, rad
ω_1 = rotational speed of journal, rad/sec
ω_2 = rotational speed of the load, rad/sec
$\Omega = \omega_1 - 2\omega_2$, rad/sec
ϵ = eccentricity ratio, e/c
o = subscript denoting initial value
· = differentiation with respect to time

In 1920, Harrison (1)[2] made the first dynamic analysis of the journal under the influence of external load and fluid pressure forces. His classical work results in two well-known equations of motion [(refer to equations (1) and (2)] from which important fundamental information has been derived. Swift (2) and Burwell (3) applied Harrison's results to several important examples, and in 1933 Robertson (4) derived equations of motion that included certain previously neglected second-order terms. In 1949 Shaw and Macks (5) gave a rather complete up-to-date summary of the subject, as well as an extensive bibliography.

[2] Underlined numbers in parentheses designate References at the end of the paper.

The purpose of this paper is to analyze Harrison's equations by using phase-plane methods (resulting in velocity versus displacement plots of the journal eccentricity) and by this treatment acquire greater insight into journal-bearing behavior. It is believed that this type of analysis exhibits previously known journal-bearing characteristics more clearly and introduces an explanation of certain other characteristics which were previously obscure.

For example, all possible motions of the journal that can result from varying such parameters as journal rotational speed, load, eccentricity, and so on, are clearly shown. Thus, the sensitivity of the journal to the half-frequency instability is brought out by the insight gained in studying the effects on the phase-plane plots of varying these parameters.

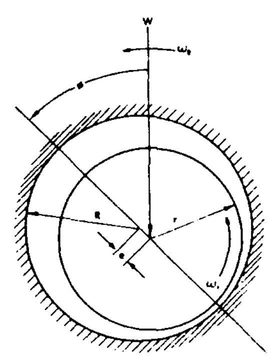


Fig. 1 Journal bearing configuration relative to fixed point in space

DESCRIPTION OF GENERAL PROBLEM

Equations of Motion

The system considered in this paper is illustrated in Fig.1. The dynamic equivalent of this system, shown in Fig.2, regards the load as fixed and both the bearing and journal as rotating.

The perfectly aligned, infinitely long, massless journal rotating in a complete bearing while under the influence of fluid pressure and external load forces was analyzed by Harrison who de-

rived the equations (reference 5, p. 226)

$$\cos \phi = \frac{J \frac{d\epsilon}{dt}}{(1 - \epsilon^2)^{3/2}} \qquad (1)$$

$$\sin \phi = \frac{J\epsilon \left(\Omega - 2 \frac{d\phi}{dt}\right)}{(2 + \epsilon^2) \sqrt{1 - \epsilon^2}} \qquad (2)$$

where the eccentricity ratio $\epsilon = e/c$, ϕ is the attitude angle measured between the load and eccentricity vectors, $\Omega = \omega_1 - 2\omega_2$ and J is the constant

$$J = \frac{12 \pi \mu r^3 L}{W c^2} \qquad (3)$$

$J\omega_1 = 12\pi^2 S$ where S is the Sommerfeld number. When ϕ varies with time, $d\phi/dt$ is the whirl speed (see Nomenclature for the definition of other terms).

If equations (1) and (2) could be solved simultaneously, the journal motion would be determined by the solution functions $\phi(t)$ and $\epsilon(t)$. Since this does not seem possible, two alternative procedures would be to either consider the four-dimensional phase space in the displacements and velocities of the two variables, or to consider simultaneously two separate phase planes.

As the first step towards the latter approach, equations (1) and (2) may be combined and ϕ eliminated to make possible the derivation of the phase plane equations for the ϵ variable (see Appendix 1), but it does not seem possible to eliminate ϵ between equations (1) and (2) in order to form the phase-plane equations for the ϕ variable. However the solution $\phi = \phi(\epsilon)$ may be obtained by eliminating dt between equations (1) and (2) (reference 5, p. 358), which would make possible a point-by-point construction of the ϕ phase plane from the ϵ phase plane.

In general, if the ϵ phase-plane nonordinary points are first determined, it is possible to gain important information about the complete system by examining the behavior of the derivatives $d\phi/dt$ and $d^2\phi/dt^2$ at those points. As will be seen later, this general approach is simple to carry out for this particular system.

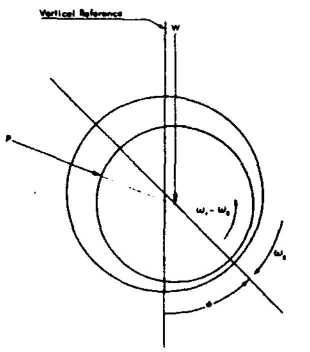


Fig. 2 Journal bearing configuration relative to fixed point on the load vector

Singular, Critical, and Equilibrium Points

Since the procedure followed here will be to describe the journal motion from concurrent investigations of the ϵ phase plane and equations (1) and (2), it is essential to distinguish between a mathematical singular point, a critical point, and a journal equilibrium point. This is because neither a singular point nor a critical point for the one-degree-of-freedom ϵ phase plane is necessarily an equilibrium point for the two-degree-of-freedom journal motion. For other phenomenon completely describable by one-degree-of-freedom, critical points are equilibrium points.

Possible indefiniteness can also occur if it is not realized that although insight useful for sketching the phase plane "integral curves" (geometrical configurations) may be achieved by investigating the mathematical singular points of the phase-plane equation written as

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)} \qquad (4)$$

some of these points may not even be critical points of the system written as

$$\dot{x} = \frac{dx}{dt} = y \qquad \dot{y} = \frac{dy}{dt} = \Phi(x, y) \qquad (5)$$

which define the phase-plane "trajectories." In this terminology the integral curves of equation (4) are the same as the trajectories of equation (5) but lack their "time sense" (reference 6, p. 116). The term "singular point" will be used to designate a nonordinary point on an integral curve while "critical point" will refer to such a point on a trajectory. Each critical point of equation (5) is usually a singular point of equation (4). In all cases the journal's equilibrium points are only at the locations where

$$d\epsilon/dt = d^2\epsilon/dt^2 = d\phi/dt = d^2\phi/dt^2 = 0$$

If the following derivative expressions are formed from equations (1) and (2), the exact location of the equilibrium points will be made clear:

$$\frac{d\epsilon}{dt} = \frac{1}{J}(1 - \epsilon^2)^{3/2} \cos\phi$$

$$\frac{d\phi}{dt} = \frac{\Omega}{2} - \sin\phi \frac{(2 + \epsilon^2)}{2J\epsilon} \sqrt{1 - \epsilon^2}$$

$$\frac{d^2\epsilon}{dt^2} = -\frac{1}{J}\left[(1 - \epsilon^2)^{3/2} \sin\phi \frac{d\phi}{dt} + 3\epsilon(1 - \epsilon^2)^{\frac{1}{2}} \cos\phi \frac{d\epsilon}{dt}\right]$$

$$\frac{d^2\phi}{dt^2} = -\frac{1}{2J}\left[\frac{(2+\epsilon \qquad \cdot \epsilon \cdot ^2}{} \cos\phi \frac{d\phi}{dt} - \left(\frac{2\epsilon^4 - \epsilon^2 + 2}{\epsilon^2 (1-\epsilon^2)^{1/2}}\right) \sin\phi \left(\frac{d\epsilon}{dt}\right)\right] \qquad (6)$$

Equations (6) show that only the point located at $\phi = \pi/2$, and the particular value of ϵ which makes $d\phi/dt = o$ is an equilibrium point for the system. At $\epsilon^2 = 1$ all derivatives are zero except $d\phi/dt$, which is then equal to $\Omega/2$. This is the half-frequency whirl phenomena which will be the instability encountered at that ϵ phase plane critical point.

Phase-Plane Equations

The differential equation defining the radial motion of the journal center is

$$\frac{d^2\epsilon}{dt^2} + \frac{(7\epsilon^2 + 2)}{2\epsilon(1-\epsilon^2)}\left(\frac{d\epsilon}{dt}\right)^2 + \frac{\Omega}{2J}\sqrt{(1-\epsilon^2)^3 - \left(J\frac{d\epsilon}{dt}\right)^2} - \frac{(1-\epsilon^2)^2(2+\epsilon^2)}{2\epsilon J^2} = 0 \qquad (7)$$

where, in accordance with the notation introduced in Appendix 1, $0 \leq \phi \leq \pi$ and ϵ is <u>positive</u> when the journal center is to the right of the fixed vertical reference in Fig.2, and <u>negative</u> when the journal center is to the left of the vertical reference line. It is understood that this notation is a mathematical convenience only, since a negative ϵ has no physical meaning.

Equation (7) may be put in conventional, nondimensional phase-plane notation and form by letting $\epsilon = x$ where $-1 \leq x \leq +1$ and $J\, d\epsilon/dt = y$. The equations defining the ϵ phase-plane trajectories are then

$$\dot{x} = \frac{dx}{dt} = \frac{1}{J} y$$

$$\dot{y} = \frac{dy}{dt} = -\frac{(7x^2 + 2)}{2x(1-x^2)}\frac{y^2}{J} - \frac{\Omega}{2}\sqrt{(1-x^2)^3 - y^2} + \frac{(1-x^2)^2(2+x^2)}{2xJ} \qquad (8)$$

and the equation defining the ϵ phase-plane integral curves may be put in the following form:

$$\frac{dy}{dx} = \frac{-(7x^2+2)y^2 - J\Omega x(1-x^2)\sqrt{(1-x^2)^3 - y^2} + (1-x^2)^3(2+x^2)}{2xy(1-x^2)} \equiv \frac{Y(x\ y)}{X(x,y)} \qquad (9)$$

All integral curves in the phase plane as defined by equation (9) will be symmetrical about the x-axis, but the x in the term containing the square root destroys the symmetry about the y-axis.

Evaluation of Nonordinary Points

Critical points in the ϵ phase plane are values of (x,y) that make $\dot{x} = \dot{y} = o$ in equations

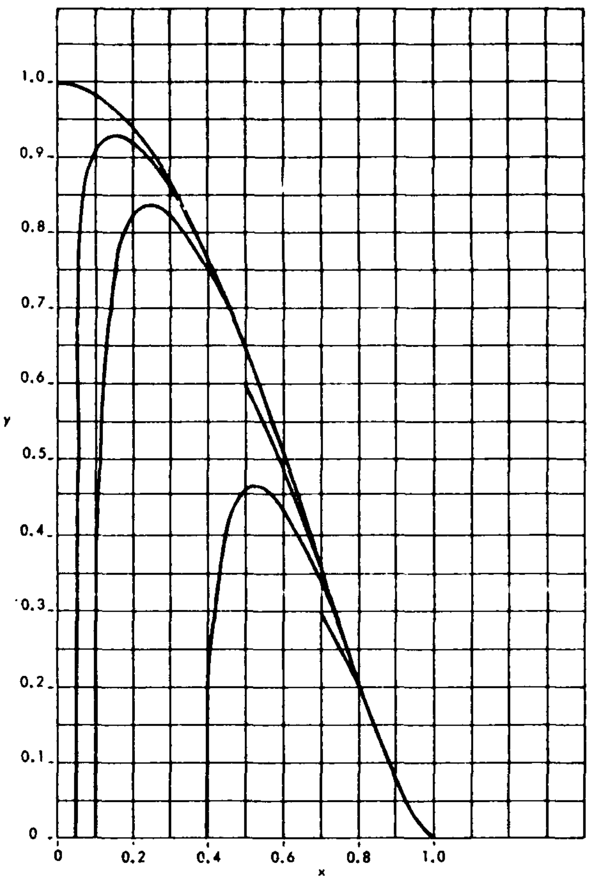
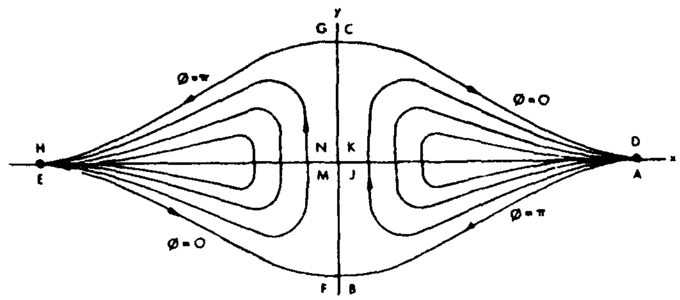


Fig. 3 Ph   plot for $\Omega = 0$

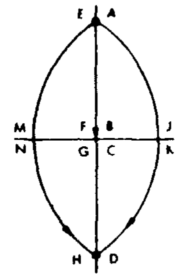
(8). The singular points of the integral curves are at the locations which simultaneously make $X(x,y)$ and $Y(x,y) = 0$ in equation (9). The coordinates of either type of point are denoted by (x^*,y^*). The co-ordinates measured from such a point are distinguished by a bar. Thus,

$$\bar{x} = x - x^*$$
$$\bar{y} = y - y^* \qquad (10)$$

A procedure (outlined below, but which is explained in detail by Coddington and Levinson, reference 7) for determining the nature of a critical point is to first transform the phase-plane equations into those whose co-ordinates are measured from the critical point (x^*,y^*) by using equations (10). The system then considered is one where this transformation results in expressions of the form



(a) Phase Plane



(b) Journal Motion

Fig. 4 Configurations for $\Omega = 0$

$$\frac{d\bar{y}}{dt} = c\,\bar{x} + d\,\bar{y} + f_2(\bar{x}, \bar{y})$$
$$\frac{d\bar{x}}{dt} = a\,\bar{x} + b\,\bar{y} + f_1(\bar{x}, \bar{y}) \qquad (11)$$

where a,b,c,d are real constants, $ad - bc \neq 0$ and f_1, f_2 are real continuous functions defined within some circle about (x^*, y^*) with radius $r_o > 0$. The functions f_1 and f_2 are called perturbations and the nonlinear system described by equation (11) is referred to as the system perturbed from the linear system given by

$$\frac{d\bar{y}}{dt} = c\,\bar{x} + d\,\bar{y}$$
$$\frac{d\bar{x}}{dt} = a\,\bar{x} + b\,\bar{y} \qquad (12)$$

After the conversion of f_1 and f_2 into polar coordinates, if f_1/r_o and f_2/r_o approach zero as $r_o \to 0+$ the perturbations are then guaranteed to tend to zero faster than the linear terms in equation (11). Under these conditions the behavior of the trajectories of equation (11) near the critical point are very similar to the behavior of the trajectories of equation (12). For the two cases considered, the theorems given in (7) are referenced freely in describing the journal bearing's phase plane portraits.

THE CASE OF $\Omega = 0$

Phase Plane Portraits

For the special case of the load rotating at one half the journal speed ($\Omega = \omega_1 - 2\,\omega_2 = 0$) or, equivalently, the case of a nonrotating journal acted upon by a nonrotating load, equation (7) reduces to the following:

$$\frac{d^2\epsilon}{dt^2} + \frac{7\epsilon^2 + 2}{2\epsilon(1-\epsilon^2)}\left(\frac{d\epsilon}{dt}\right)^2 - \frac{(1-\epsilon^2)^2\,(2+\epsilon^2)}{2\epsilon J^2} = 0 \qquad (13)$$

For this case, it is more meaningful to investigate the equation for the integral curves which, from equation (13), becomes

$$\frac{dy}{dx} = \frac{-(7x^2+2)\,y^2 + (1-x^2)^3\,(2+x^2)}{2xy\,(1-x^2)} \equiv \frac{Y(x,y)}{X(x,y)} \qquad (14)$$

where now the phase-plane portrait is symmetrical about both the x and y-axes.

The singular points in equation (14) are at the four locations $(\pm 1, 0)$ and $(0, \pm 1)$. Although, for completeness, a brief discussion of these singular points will be given in the next section, their evaluation is not necessary in order to obtain the phase-plane portrait. This is because equation (13) may be integrated once to yield an explicit equation for the integral curves and, when the constant of integration is zero ($x_o = 0$), it may be integrated again to give x as a function of time. The result of the first integration (see Appendix 2) is

$$y^2 = \frac{x_o^2\left[y_o^2 - (1-x_o^2)^3\right]}{(1-x_o^2)^{9/2}}\;\frac{(1-x^2)^{9/2}}{x^2} + (1-x^2)^3 \qquad (15)$$

From equation (15) it is seen that the integral curves meeting the y-axis will be given by

$$y = \pm\,(1-x^2)^{3/2} \qquad (16)$$

Equation (16) defines the boundary of a closed area. Each boundary arc connects singular points and is called a separatrix arc. It is easily seen that all physically meaningful integral curves must lie inside the boundary defined by equation (16), for if $y > (1-x^2)^{3/2}$ or $y < -(1-x^2)^{3/2}$, then equation (1) would imply $\cos\phi > 1$ or $\cos\phi < -1$. Moreover, the difference between the ordinate of an integral curve beginning inside the separatrices and the ordinate of a separatrix itself remains finite and tends to zero only at the singular points $(\pm 1, 0)$. The integral curves are tangent to the separatrixes at $(\pm 1, 0)$, but they do not touch them at any other point. Fig.3 shows integral curves plotted from equation (15).

Under zero initial conditions the solution of equation (16) is

$$x = \frac{t}{\sqrt{J^2 + t^2}} \qquad (17)$$

which gives the change in ϵ with time along a separatrix arc (trajectory).

Examination of equations (1) and (2) reveals how ϕ varies with time along a separatrix arc. For convenience express equation (1) and (2) in phase-plane notation, thus

$$\cos \phi = \frac{y}{(1 - x^2)^{3/2}} \qquad (18)$$

$$\sin \phi = \frac{J x (\Omega - 2\frac{d\phi}{dt})}{(2 + x^2)\sqrt{1 - x^2}} \qquad (19)$$

When $\phi = 0$ and π equation (18) reduces to equation (16). When $\phi = 0$ or π equation (19) requires $d\phi/dt = 0$ for $\Omega = 0$ and $\phi < x < 1$. The separatrix arc between (0, +1) and (+1, 0) is the integral curve along which $\phi = 0$ and $d\phi/dt = 0$ while the path between (1,0) and 0, -1) requires $\phi = \pi$ and $d\phi/dt = 0$. Similar statements apply to the negative half plane.

The physical interpretation of journal behavior can be most easily explained by referring to Fig.4. As trajectory A,B,C,D in Fig.4(a) is traversed in the phase plane, the journal travels vertically down from wall contact at $\phi = \pi$ to wall contact at $\phi = 0$. For other trajectories such as A,J,K,D, the journal motion is to the right of the vertical reference as indicated in Fig.4(b). Paths are to the left of the vertical reference for trajectories such as E,M,N,H. This type of journal behavior has been previously computed and sketched by Swift (reference 5, p. 222).

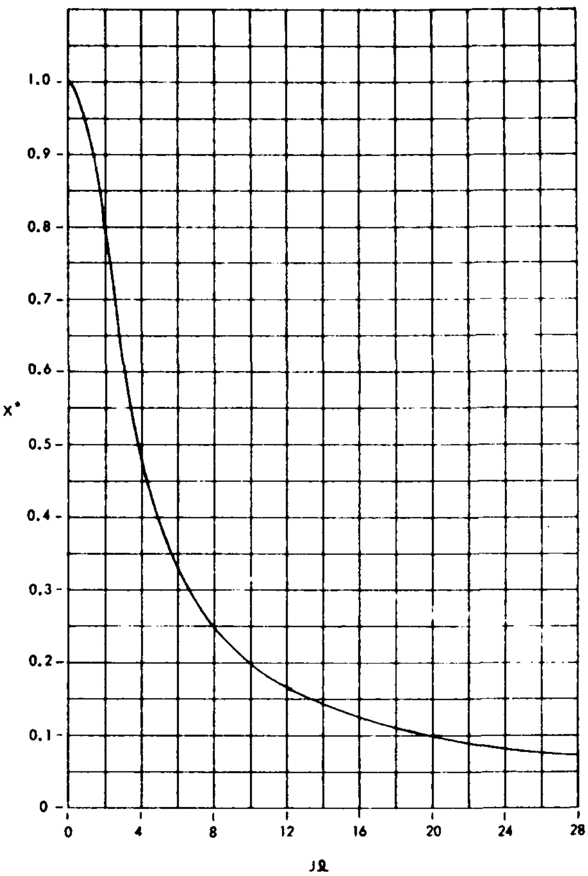


Fig. 5 Center singularity location versus J Ω

THE CASE OF $\Omega \neq 0$

Singular Points in the Phase Plane

In general, $\Omega \neq 0$ and the integral curves are defined by

$$\frac{dy}{dx} = \frac{-(7x^2 + 2)y^2 - J\Omega x(1 - x^2)\sqrt{(1 - x^2)^3 - y^2} + (1 - x^2)^3(2 + x^2)}{2xy(1 - x^2)} \equiv \frac{Y(x,y)}{X(x,y)} \qquad (9)$$

Phase-Plane Nonordinary Points

Among the four nonordinary points of equation (14), only those at ($\pm$ 1, 0) are critical points and, from equation (6), these points are equilibrium points for the system. The theorems of Coddington and Levinson (7) indicate unstable "node-like" critical points at ($\pm$ 1, 0) and "saddle like" singularities at (0, $\pm$ 1).

The case of $\Omega = 0$ is, by itself, of only slight interest, but it does serve as a foundation for the more involved case of $\Omega \neq 0$. In addition, it gives an indication of the journal behavior as it slows down for stopping (if $\omega_2 = 0$), or as the load rotational speed approaches one half the journal rotational speed.

which has singular points at (0, $\pm$ 1) and at the points along the x-axis where

$$(1 - x^{*2})^{5/2}\left[- J\Omega x^* + (1 - x^{*2})^{1/2}(2 + x^{*2})\right] = 0 \qquad (20)$$

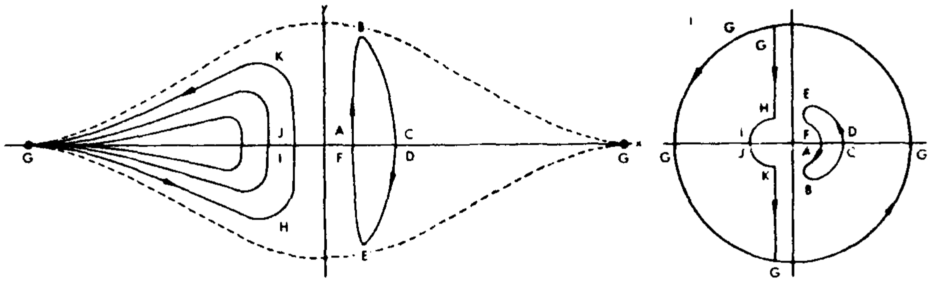
Equation (20) yields the singular points at ($\pm$ 1, 0), and a fifth singularity whose x^* location is contained in the solution of the cubic equation

$$z^3 + 3z^2 + J^2\Omega^2 z - 4 = 0 \qquad (21)$$

where $z = x^{*2}$. Consideration of equation (21) in the root locus form

$$1 + \frac{J^2\Omega^2 z - 4}{z^2(z + 3)} = 0 \qquad (22)$$

shows that it will always have a positive z root



(a) Phase Plane (b) Journal Motion

Fig. 6 Configurations for large values of $J\Omega$

located between $z = 0$ and $z = 4/J^2\Omega^2$. For values of $J\Omega \geq 6$, x^* is closely approximated by

$$x^* = 2/J\Omega \quad \text{for } J\Omega \geq 6, \tag{23}$$

where only the positive sign resulting from $Z^{1/2}$ satisfies equation (20).[3] The values of x^* satisfying equation (20) are plotted in Fig.5 which is the familiar Sommerfeld eccentricity curve defining the equilibrium position. Thus the singular points for $\Omega \neq 0$ include those for $\Omega = 0$ plus a fifth point. Before investigating the nature of these singular points, it is desirable to examine equations (18) and (19) under conditions when $\Omega \neq = 0$

$$\cos\phi = \frac{y}{(1 - x^2)^{3/2}} \tag{18}$$

$$\sin\phi = \frac{J\,x\,(\Omega - 2\frac{d\phi}{dt})}{(2 + x^2)\sqrt{1 - x^2}} \tag{19}$$

For $\phi = 0$ and π equation (18) reduces to equation (16), which from equation (9), is also an integral curve of the ϵ phase plane as in the case of $\Omega = 0$. The right side of equation (19) can then only be zero when $d\phi/dt = \Omega/2$ (for $x \neq 0$ and $\Omega \neq 0$), which implies ϕ changing with time. Therefore, the separatrix arc defined by equation (16) is no longer a physically possible continuous trajectory as in the case of $\Omega = 0$, but only points along it are possible instantaneously when $d\phi/dt$ passes through $\Omega/2$.

Determination of the Type of Singular Points

The application of the techniques described by Coddington and Levinson (7) to equations (8) reveals the following:

1 The critical points at $(\pm 1,0)$ are of higher order as was true in the case $\Omega = 0$, but after dividing by $\bar{y}$ the perturbations f_1, f_2 fail to approach zero in the required manner. Therefore, the existing theory cannot explain their type. However, it is known from equation (6) that these ϵ phase-plane critical points define the half-frequency whirl phenomenon.

2 The singularities at $(0, \pm 1)$ are the "saddle like" type identical to those at the same locations for the case of $\Omega = 0$.

3 The new critical point (also the equilibrium point) at $x^* \cong 2/J\Omega$ is a center for the linear case, and is either a spiral point or center for the nonlinear case (reference 7, p. 382, Theorem 4.1). Since equation (9) is symmetrical about the x-axis, this critical point must be a center for the nonlinear case also. A center is symmetrical -- a spiral point is not. Closed trajectories indicating neutral stability will surround this center in the phase plane. Thus, if the journal were disturbed slightly, the resulting motion would be continuously oscillatory, corresponding to the case in a linear system of conjugate complex roots on the imaginary axis.

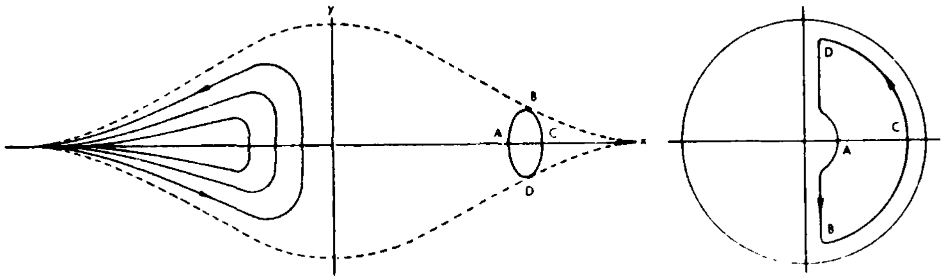
Journal Behavior for Various Values of $J\Omega$

If equation (19) is divided by $J\Omega$, and $J\Omega$ is considered large, the conclusion is that x must be small in order to satisfy the resulting equation, provided $(2/\Omega)$ $(d\phi/dt)$ may be neglected in comparison with one. For small values of x, equation (9) becomes

$$\frac{dy}{dx} = \frac{-2y^2 - J\Omega x\sqrt{1 - y^2} + 2}{2xy} \tag{24}$$

Equation (24) has singular points at $(0, \pm 1)$ and $(2/J\Omega\ 0)$, which agrees with the results found previously near $x = 0$. Moreover, the singularity at $x = 2/J\Omega$ is a center, and trajectories around the center reach their absolute y value when $dy/dx = 0$ in equation (24). This occurs at $y = \pm 1$.

[3] If Ω were negative, only the negative x^* would satisfy equation (20), but only positive Ω-values are considered in this analysis. If the effective rotational speed of the journal were in the opposite direction, the results would be symmetrical.



(a) Phase Plane (b) Journal Motion

Fig. 7 Configurations for small values of J Ω

Near $y = \pm 1$, ϕ is near 0 and π, respectively. For high values of $J\Omega$ the journal, if disturbed, whirls at very small eccentricity ratios with attitude angles continuously changing and approaching very close to zero and π.

Fig.6(a) shows a sketch of the phase-plane portrait taken from IBM 650 computer solutions of equation (9). The left half plane of Fig.6(a) is similar to that for the case of $\Omega = 0$. As trajectory A,B,C,D,E,F is traversed in Fig.6(a), the corresponding physical journal motion is pictured in Fig.6(b). When the journal is located at either B or E, any slight disturbance due to forces not considered in this analysis could cause the journal to jump into a trajectory leading toward the half-frequency whirl singularity at G.

Possible confusion as to the interaction of the negative and positive half-phase planes in Fig.6(a) may be clarified by visualizing the negative half-plane folded about the vertical y-axis onto the positive half-plane. This type of mental image is important when considering cases where trajectories pass close to (+1, 0) due to, for example, low values of $J\Omega$. Fig.7 is similar in interpretation to Fig.6 with the exception that the possibility of travel to the unstable singularity occurs at an eccentricity ratio near 1.

With the aid of equation (3), it is seen that large values of $J\Omega$ may occur because of high effective journal speeds and/or light journal loads. Other factors influence J, such as viscosity, c/r, and so on. It is also worth while to note that since the journal attitude angle must reach 0 or π for the journal to pass into the unstable left half-phase region, a journal that possesses both large J (light loading) and high Ω is in a particularly vulnerable position. This is because $\phi = 0, \pi$ at $y \equiv J(d\epsilon/dt) = \pm 1$, and for high J the eccentricity velocity necessary for $\phi = 0, \pi$ is small. This deduction gives important insight into journal bearing stability.

SUMMARY AND CONCLUSIONS

On the basis of the assumptions made in this analysis, it appears possible to conclude the following:

1 Complete journal bearings with continuous films subjected only to external load and fluid pressure forces are at best neutrally stable.

2 For journal bearings operating at high effective rotational speeds or large values of J (light loadings), the likelihood of unstable motion appears strong if the journal is acted upon by a sudden disturbance. This is especially true when the combination of both high speed and large J-values exist.

3 Unstable motion of the journal bearing can also occur at very low values of $J\Omega$.

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APPENDIX 1

Equations (1) and (2) are of the form:

$$\cos \phi = f \frac{d\epsilon}{dt} \tag{25}$$

$$\sin \phi = g - k \frac{d\phi}{dt} \tag{26}$$

where f, g, and k are functions of ϵ and constants. Before combining equations (25) and (26) and eliminating ϕ, it is convenient to restrict ϕ to the interval $0 \leq \phi \leq \pi$ and allow ϵ to take on any of the values in the interval $-1 \leq x \leq 1$. To show that equations (1) and (2) are unchanged by this restriction, define

$$\phi' = \begin{cases} \phi & \text{when } 0 \leq \phi \leq \pi \\ \phi - \pi & \text{when } \pi \leq \phi \leq 2\pi \end{cases}$$

$$\epsilon' = \begin{cases} \epsilon & \text{when } 0 \leq \phi \leq \pi \\ -\epsilon & \text{when } \pi \leq \phi \leq 2\pi \end{cases}$$

where $0 \leq \phi \leq 2\pi$ and $0 \leq \epsilon \leq 1$. If $0 \leq \phi \leq \pi$, equations (1) and (2) are clearly unchanged. If $\pi \leq \phi \leq 2\pi$, then $\cos \phi' = -\cos \phi$, $\sin \phi' = -\sin \phi$, $d\epsilon'/dt = -d\epsilon/dt$, $d\phi'/dt = d\phi/dt$, and $\epsilon' = -\epsilon$. Substitution of these into equations (1) and (2) yields equations in ϕ' and ϵ' which are of the same form. The advantage of this transformation is that $\sin \phi' = +(1 - \cos^2 \phi')^{1/2}$ and the ambiguous minus sign is eliminated.

From equation (25) $\sin \phi$ is derived from the right triangle which gives

$$\sin \phi = \left[1 - \left(f \frac{d\epsilon}{dt}\right)^2\right]^{1/2} \tag{27}$$

If equation (25) is differentiated with respect to time, there results

$$-\sin \phi \frac{d\phi}{dt} = f \frac{d^2\epsilon}{dt^2} + \frac{d\epsilon}{dt} \frac{df}{dt} \tag{28}$$

Multiplying both sides of equation (26) by $\sin \phi$ gives

$$\sin^2 \phi = g \sin \phi - k \sin \phi \frac{d\phi}{dt} \tag{29}$$

Substitution of equation (27) and (28) into equation (29) and rearranging terms gives

$$k f \frac{d^2\epsilon}{dt^2} + k \frac{df}{dt} \frac{d\epsilon}{dt} + f^2 \left(\frac{d\epsilon}{dt}\right)^2 + g \sqrt{1 - \left(f\frac{d\epsilon}{dt}\right)^2} - 1 = 0 \tag{30}$$

After the indicated differentiation and substitutions are performed, equation (30) reduces to equation (7).

APPENDIX 2

Equation (13) is

$$\frac{d^2\epsilon}{dt^2} + \frac{(7\epsilon^2 + 2)}{2\epsilon(1 - \epsilon^2)} \left(\frac{d\epsilon}{dt}\right)^2 - \frac{(1 - \epsilon^2)^2 (2 + \epsilon^2)}{2\epsilon J^2} = 0 \tag{13}$$

Let

$$\left(\frac{d\epsilon}{dt}\right)^2 = \dot{\epsilon}^2 = v,$$

Then

$$\frac{d^2\epsilon}{dt^2} = \frac{1}{2} \frac{d\dot{\epsilon}^2}{d\epsilon} = \frac{1}{2} \frac{dv}{d\epsilon}$$

Equation (13) becomes

$$\frac{dv}{d\epsilon} + \frac{(7\epsilon^2 + 2)}{\epsilon(1 - \epsilon^2)} v = \frac{(1 - \epsilon^2)^2 (2 + \epsilon^2)}{\epsilon J^2} \tag{31}$$

which is a linear differential equation in v whose solution, placed in phase-plane notation, reduces to equation (15).